Feb 9, 1995

TAUP–2234–95

hep-th/9502058

On Topological $2D$ String and Intersection Theory

Yaron Oz*

yarono@ccsg.tau.ac.il

School of Physics and Astronomy

Raymond and Beverly Sackler Faculty of Exact Sciences

Tel-Aviv University

Ramat Aviv, Tel-Aviv 69978, Israel.

Abstract

The topological description of $2D$ string theory at the self-dual radius is studied in the algebro-geometrical formulation of the A_{k+1} topological models at $k = -3$. Genus zero correlators of tachyons and their gravitational descendants are computed as intersection numbers on moduli space and compared to $2D$ string results. The interpretation of negative momentum tachyons as gravitational descendants of the cosmological constant, as well as modifications of this, is shown to imply a disagreement between $2D$ string correlators and the associated intersection numbers.

*Work supported in part by the US-Israel Binational Science Foundation, and GIF - the German-Israeli Foundation for Scientific Research.

$2d$ string theory at the self-dual radius is described by a topological field theory with an infinite number of primary fields. A topological Landau-Ginzburg formulation of the theory has been constructed in [1, 2], and a topological interpretation of the $W_{1+\infty}$ recursion relations for the tachyon correlators has been given in [3, 4].

However, the topological structure of the theory is still not well understood. In particular, the identification of the BRST cohomology states as gravitational primaries and descendants, and the relation between the correlators and intersection numbers on the moduli space of Riemann surfaces has not been clarified yet.

It has been suggested in [1, 2] that while non-negative momentum tachyons correspond to primary fields, the negative momentum tachyons correspond to gravitational descendants of T_0, the zero momentum tachyon. $2d$ string theory, however, is parity invariant under the reflection of momenta $k \leftrightarrow -k$, thus positive and negative momentum tachyons appear symmetrically. This might imply that all the tachyons should be considered as primaries, which is not in contradiction with the formulations of [1, 3]. Such an interpretation is in agreement with intersection calculations of the four-point tachyon correlator [5, 6] as well as the $1 \to n$ tachyon amplitude and five-point function at various kinematical regions [7].

There is a third possibility, namely that the negative momentum tachyons can be considered both as primaries and as descendants. This, if correct, will lead to beautiful relations between intersection numbers on the moduli space of Riemann surfaces, and should be intimately related to the way gravitational descendants are built from matter [8].

Since the generating functional for tachyon correlators is a tau function of the Toda lattice hierarchy [9], one expects a generalization of the relation that exists between intersection numbers and the KdV hierarchy as implied by topological gravity [10, 11].

In this letter we make a step in the study of the intersection theory associated with the topological $2d$ string. In particular, we study the possibility of relating negative momentum tachyons to gravitational descendants from the viewpoint of intersection theory on the moduli space of Riemann surfaces. We compute tachyon correlators as intersection numbers and show that the identification of negative momentum tachyons as gravitational descendants, as well as modifications of this, is in disagreement with $2d$ string results.

The algebro-geometrical set up is given by the analytical continuation of that, associated with the A_{k+1} theory [5, 12], to $k = -3$. Define the "vector bundles" ν and ω over the compactified moduli space of genus g Riemann surfaces with s punctures $\bar{\mathcal{M}}_{g,s}$, whose

fibers V and W respectively are [†]:

$$V = H^0(\Sigma_{g,s}, \mathcal{R}), \qquad W = H^1(\Sigma_{g,s}, \mathcal{R}) \ , \tag{1}$$

where $\mathcal{R} \equiv K^2 \otimes_i O(z_i)^{1-k_i}$ is a line bundle over $\Sigma_{g,s}$, K being the canonical bundle. The sections of $\mathcal{R}$ are two-forms that can have zeros at the points z_i with order of at least $k_i - 1$.

Then

$$\langle \prod_{i=1}^{s} \sigma_{n_i}(T_{k_i}) \rangle_g = \int_{\bar{\mathcal{M}}_{g,s}} \prod_{i=1}^{s} (C_1(\mathcal{L}_i))^{n_i} C_T(\nu \ominus \omega) \ , \tag{2}$$

where $\sigma_n(T_k)$ is the n-th descendant of the momentum k tachyon, C_T is the top Chern class and $C_1(\mathcal{L}_i)^n \in H^{2n}(\bar{\mathcal{M}}_{g,s})$ are the Mumford-Morita-Miller stable classes. $\mathcal{L}_i$ is a line bundle on the moduli space with fiber at Σ being the cotangent space to the point $x_i \in \Sigma$. The subtraction of the vector bundles ν and ω should be understood in $K-$theory sense.

Using the Riemann-Roch theorem for a line bundle L

$$dim H^0(\Sigma_g, L) - dim H^1(\Sigma_g, L) = deg L - g + 1 \ , \tag{3}$$

and $deg(K) = 2g - 2$, we get the selection rule for (2)

$$\sum_{i=1}^{s} (n_i - k_i) = 0 \ . \tag{4}$$

Let us consider from the intersection theory viewpoint, the definition of negative momentum tachyons T_{-n} as the n-th gravitational descendants of T_0, $T_{-n} = \sigma_n(T_0)$. The first family of correlators we wish to evaluate is $\langle \prod_{i=1}^{s} \sigma_{n_i}(T_{k_i}) \rangle_0$ subject to the constraint $\sum_{i=1}^{s} n_i = s-3$ and satisfying (4). Since $deg(\mathcal{R}) = -1 < 0$, the Kodaira vanishing theorem implies that $dim V = 0$. Using the Riemann-Roch theorem (3) we get that $dim W = 0$. Thus, in this case the correlator is given by a simple integral

$$\langle \prod_{i=1}^{s} \sigma_{n_i}(T_{k_i}) \rangle_0 = \int_{\bar{\mathcal{M}}_{0,s}} \prod_{i=1}^{s} (C_1(\mathcal{L}_i))^{n_i} = \frac{(s-3)!}{\prod_{i=1}^{s} n_i!} \ . \tag{5}$$

This result is in contradiction with T_{-n} being defined as the n-th gravitational descendants of T_0, since the correlators of $c = 1$ tachyons satisfy

$$\langle \prod_{i=1}^{s} T_{k_i} \rangle_0 = 0 \quad if \quad \sum_{i=1}^{s} |k_i| \Theta(-k_i) = s - 3 \ , \tag{6}$$

[†] ν and ω are called the (higher) direct image sheaves of $\mathcal{R}$ and are not necessarily locally free, i.e. the dimension of the fiber may depend on the point on $\bar{\mathcal{M}}_{g,s}$.

which is proved using the T_0 and T_1 Ward identities of the theory.

Note, however, that it is possible to overcome the disagreement between (5) and (6) by postulating an infinite factor between the tachyon T_{-n} and the gravitational descendant $\sigma_n(T_0)$.

Consider next the genus zero three point function

$$\langle\prod_{i=1}^{3}\sigma_{n_i}(T_{k_i})\rangle_0 = \int_{\bar{\mathcal{M}}_{0,3}}\prod_{i=1}^{3}(C_1(\mathcal{L}_i))^{n_i}C_T(\nu\ominus\omega)\ , \tag{7}$$

where $\sum_{i=1}^{3}(n_i-k_i)=0$. $deg(\mathcal{R})=-\sum_{i=1}^{3}k_i<0$, thus $dimV=0$ and $dimW=\sum_{i=1}^{3}k_i$. The integral in (7) is evaluated via

$$\begin{aligned}\int_{\bar{\mathcal{M}}_{0,3}}\prod_{i=1}^{3}(C_1(\mathcal{L}_i))^{n_i}C_T(-\omega) &= \int_{\bar{\mathcal{M}}_{0,3}}\prod_{i=1}^{3}C_T(\mathcal{L}_i^{\oplus n_i})C_T(-\omega) = \\ &= \int_{\bar{\mathcal{M}}_{0,3}}C_T(\mathcal{L}_{grav}\ominus\omega) = \int_{\bar{\mathcal{M}}_{0,3}}C_0(\mathcal{L}_{grav}\ominus\omega) = 1\ ,\end{aligned} \tag{8}$$

where $\mathcal{L}_{grav}=\oplus_{i=1}^{3}\mathcal{L}_i^{\oplus n_i}$, and we used a basic property of the top Chern class of a Whitney sum of vector bundles

$$C_T(\nu\oplus\omega)=C_T(\nu)C_T(\omega)\ . \tag{9}$$

Equation (8) is in agreement with the three point tachyon correlator

$$\langle T_{k_1}T_{k_2}T_{k_3}\rangle = \delta_{k_1+k_2+k_3,0}\ , \tag{10}$$

thus excluding the possibility of an infinite factor between the negative momentum tachyon and the gravitational descendant. Note that in contrast to (8), in minimal topological models coupled to topological gravity three point functions with gravitational descendants vanish, which is one of the basic ingredients of the topological recursion relations of [5].

Consider now the genus zero four point function

$$\langle\prod_{i=1}^{4}\sigma_{n_i}(T_{k_i})\rangle_0 = \int_{\bar{\mathcal{M}}_{0,4}}\prod_{i=1}^{4}(C_1(\mathcal{L}_i))^{n_i}C_T(\nu\ominus\omega)\ . \tag{11}$$

$deg(\mathcal{R})=-\sum_{i=1}^{4}k_i<0$, thus $dimV=0$ and $dimW=\sum_{i=1}^{4}k_i-1$. The integral in (11) can be recast as

$$\begin{aligned}\int_{\bar{\mathcal{M}}_{0,4}}\prod_{i=1}^{4}(C_1(\mathcal{L}_i))^{n_i}C_T(-\omega) &= \int_{\bar{\mathcal{M}}_{0,4}}\prod_{i=1}^{4}C_T(\mathcal{L}_i^{\oplus n_i})C_T(-\omega) = \\ &= \int_{\bar{\mathcal{M}}_{0,4}}C_T(\mathcal{L}_{grav}\ominus\omega) = \int_{\bar{\mathcal{M}}_{0,4}}C_1(\mathcal{L}_{grav}\ominus\omega)\ ,\end{aligned} \tag{12}$$

where $\mathcal{L}_{grav} = \oplus_{i=1}^{4} \mathcal{L}_i^{\oplus n_i}$.

In order to evaluate (12) we will use the property that the Chern character of a difference of vector bundles satisfies

$$ch(E \ominus F) = ch(E) - ch(F) \ . \tag{13}$$

The Chern Character has an expansion in terms of Chern classes

$$ch(E) = rankE + C_1(E) + \frac{1}{2}(C_1(E)^2 - 2C_2(E)) + ... \ . \tag{14}$$

Using (13) and (14) we have

$$ch(\mathcal{L}_{grav} \ominus \omega) = \sum_{i=1}^{4} n_i(1 + C_1(\mathcal{L}_i)) - (\sum_{i=1}^{4} k_i - 1 + C_1(\omega) + ...) \ . \tag{15}$$

On the other hand, since $\mathcal{L}_{grav} \ominus \omega$ is a line bundle we get

$$ch(\mathcal{L}_{grav} \ominus \omega) = 1 + C_1(\mathcal{L}_{grav} \ominus \omega) \ . \tag{16}$$

Equations (11),(12),(15) and (16) yield

$$\begin{aligned} \langle \prod_{i=1}^{4} \sigma_{n_i}(T_{k_i}) \rangle_0 &= \int_{\bar{\mathcal{M}}_{0,4}} C_1(\mathcal{L}_{grav} \ominus \omega) = \\ = \sum_{i=1}^{4} n_i \int_{\bar{\mathcal{M}}_{0,4}} C_1(\mathcal{L}_i) - \int_{\bar{\mathcal{M}}_{0,4}} C_1(\omega) &= \sum_{i=1}^{4} n_i - \int_{\bar{\mathcal{M}}_{0,4}} C_1(\omega) \ . \end{aligned} \tag{17}$$

Thus, in order to evaluate the genus zero four-point function of tachyon's gravitational descendants we have to compute the first Chern class of the vector bundle ω.

Recall now that the four point tachyon correlator is [13]

$$\langle T_k T_{k_1} T_{k_2} T_{k_3} \rangle = (k-1) - \sum_{i=1}^{3} (k + k_i)\Theta(-k - k_i) \ , \tag{18}$$

with $k > 0$. As seen in (18) the $2 \to 2$ amplitude $\langle T_{-k_1} T_{-k_2} T_{k_3} T_{k_4} \rangle_0$ is non-trivial and depends on various kinematical regions. We will now compute this amplitude from intersection theory viewpoint with negative momentum tachyons considered as gravitational descendants of T_0.

Using (17) we have

$$\langle \sigma_{k_1}(T_0) \sigma_{k_2}(T_0) T_{k_3} T_{k_4} \rangle_0 = k_1 + k_2 - \int_{\bar{\mathcal{M}}_{0,4}} C_1(\omega) \ . \tag{19}$$

In order to compute the first Chern class of the vector bundle ω, recall that

$$C_1(\omega) \equiv C_1(det\ \omega)\ , \tag{20}$$

with $det\ \omega$ being the determinant line bundle associated with ω. The rank of ω is $r = k_3 + k_4 - 1$ and its fiber is the vector space

$$\begin{aligned} &H^1(\Sigma_{0,4}, K^2 \otimes_{i=1}^2 O(z_i)^1 \otimes_{i=3}^4 O(z_i)^{1-k_i}) = \\ &H^0(\Sigma_{0,4}, K^{-1} \otimes_{i=1}^2 O(z_i)^{-1} \otimes_{i=3}^4 O(z_i)^{k_i-1})^*\ , \end{aligned} \tag{21}$$

where we used Serre duality

$$H^1(\Sigma_g, L) \cong H^0(\Sigma_g, K \otimes L^{-1})^*, \tag{22}$$

and $*$ denotes the dual bundle.

A basis of $SL(2,C)$ invariant sections of the dual bundle to ω is, for instance,

$$\begin{aligned} s_1 &= dz^{-1}\frac{(z-z_1)(z-z_2)}{z_{12}} \\ s_i &= dz^{-1}\frac{(z-z_1)^i(z-z_2)z_{13}^{i-1}}{(z-z_3)^{i-1}z_{12}^i} \qquad i = 2...k_3 \\ s_{k_3+i-1} \equiv s'_i &= dz^{-1}\frac{(z-z_1)^i(z-z_2)z_{14}^{i-1}}{(z-z_4)^{i-1}z_{12}^i} \qquad i = 2...k_4\ , \end{aligned} \tag{23}$$

where $z_{ij} \equiv z_i - z_j$. The first Chern class of ω is evaluated by counting with multiplicities the zeros minus the poles of a section $det(S)$ of $det\ \omega$. S is the $r \times r$ matrix : $S_{ij} \equiv s_i(x_j)$.

The zeros and poles of $det(S)$ are located on the boundary of $\bar{\mathcal{M}}_{0,4}$. We denote by Σ_{ij} the degenerate sphere corresponding to the merge of the punctures i and j, which in the Deligne-Mumford compactification is represented by a splitting of a sphere.

The sections s_i have a pole of order i on Σ_{12}, and a zero of order $i-1$ on Σ_{13}, while s'_i have a pole of order i on Σ_{12} and a zero of order $i-1$ on Σ_{14}, thus contributing k_3+k_4-1 to the first Chern class of ω, where we used the fact that $C_1(\omega) = -C_1(\omega^*)$.

We also have to calculate the zeros of the determinant due to linear dependencies among the sections on Σ_{ij}. Define

$$u(z) = \frac{z-z_1}{z-z_3}, \qquad v(z) = \frac{z-z_1}{z-z_4}\ , \tag{24}$$

and consider the matrix T_{ij}

$$\begin{aligned} T_{ij} &\equiv u^i(x_j) \qquad i = 0...k_3-1, \qquad j = 1...k_3+k_4-1 \\ &\equiv v^{i-k_3+1}(x_j) \quad i = k_3...k_3+k_4-2, \quad j = 1...k_3+k_4-1\ . \end{aligned} \tag{25}$$

The matrix T_{ij} is basically S_{ij}, but we factored out overall non relevant terms. It encodes the linear dependencies among the sections, and we have to calculate the zeros of its determinant on the degenerate surfaces Σ_{ij}. Consider, for instance, the degenerate surface Σ_{13}. From (24) it follows that

$$u = \frac{\alpha}{1+\frac{\varepsilon}{v}} \ , \tag{26}$$

where $\alpha = \frac{z_{14}}{z_{34}}$ and $\varepsilon = \frac{z_{13}}{z_{34}}$. We have to consider the order of zero of T_{ij} as ε goes to zero. It is convenient to change a basis from $u^i, i = 0...k_3 - 1$ to $(\frac{\varepsilon}{v})^i$. Thus, the order of zero of $det(T)$ is $\sum_{i=1}^{k_3-1} = \frac{1}{2}k_3(k_3-1)$. Similarly we get the contributions from the other degenerate surfaces. We find

$$det(T) \sim z_{13}^{\frac{1}{2}k_3(k_3-1)} z_{14}^{\frac{1}{2}k_4(k_4-1)} z_{34}^{(k_3-1)(k_4-1)} \ . \tag{27}$$

Altogether we have

$$\int_{\bar{\mathcal{M}}_{0,4}} C_1(\omega) = -\frac{1}{2}(k_3+k_4)(k_3+k_4-5) - 2 \ . \tag{28}$$

Using (19) and (28) we get

$$\langle \sigma_{k_1}(T_0)\sigma_{k_2}(T_0)T_{k_3}T_{k_4}\rangle_0 = k_1 + k_2 + 2 + \frac{1}{2}(k_3+k_4)(k_3+k_4-5) \ , \tag{29}$$

where $\sum_{i=1}^{4} k_i = 0$. For the special case $\langle\sigma_1(T_0)\sigma_1(T_0)T_1T_1\rangle_0$ only the section s_1 in (23) is relevant, and the value of the correlator is one, in agreement with (29).

The result (29) is clearly different from (18), in particular is does not depend on kinematical regions and it is not parity invariant.

Note, in contrast, that if the negative momentum tachyons are considered as primaries, then $dimV = 1, dimW = 0$, and the intersection calculation

$$\langle\prod_{i=1}^{4} T_{k_i}\rangle_0 = \int_{\bar{\mathcal{M}}_{0,4}} C_1(\nu) \ , \tag{30}$$

agrees with (18) [5].

In [4] a definition of a negative momentum tachyon as a sum of gravitational descendants in $1 \to n$ amplitudes has been proposed

$$T_{-n} = \sum_{i=0}^{n}\prod_{j=1}^{i}(j-n)\sigma_i(T_{-n+i}) \ . \tag{31}$$

This definition together with (2) is clearly compatible with the $1 \to 3$ tachyons correlator. However, for the $2 \to 2$ amplitude we get $dimV = dimW = 0$, thus

$$\langle T_{-k_1}T_{-k_2}T_{k_3}T_{k_4}\rangle_0 = (1-k_1)\int_{\bar{\mathcal{M}}_{0,4}} C_1(\mathcal{L}_1) + \\ +(1-k_2)\int_{\bar{\mathcal{M}}_{0,4}} C_1(\mathcal{L}_2) = 2 - k_1 - k_2, \tag{32}$$

in disagreement with (18).

The intersection theory computations presented above imply that the identification of negative momentum tachyons as gravitational descendants in the topological formulation of the $2d$ string leads to a disagreement between tachyon correlators and the associated intersection numbers. The identification of all the tachyons as gravitational primaries is consistent with all the intersection theory computations which have been done so far. These lead us to suggest that indeed the proper set up is to consider all the tachyons as gravitational primaries, and interpret accordingly the topological numbers associated with their correlators.

It should be clear, however, that our computations do not exclude the possibility that an insertion of a tachyon operator in a correlator might be equivalent to an insertion of some combination of gravitational descendants. This would simply mean that there are topological Ward identities underlying the theory.

The gravitational descendants of the tachyons, which some of their correlators have been calculated above, are likely to correspond to discrete states of $2d$ string theory. However, as discussed in [14], it is not clear what is the exact relation between gravitational descendants and discrete states since correlators of the latter have not been calculated successfully in the Liouville framework. Such a relation, when revealed, will evidently shed light on the relation between the physical $2d$ string theory and its topological formulation. Also, an underlying integrable structure generalizing the Toda lattice hierarchy is expected.

It is straightforward to apply the same manipulations which we used in order to derive (8) and (17), in order to get formulas for higher point functions of tachyons and their gravitational descendants. These formulas involve higher Chern classes of the sheaf ω. The latter is not locally free and we expect jumps in the dimension of the fiber at the boundary of the moduli space [‡], thus complicating the calculation of the intersection numbers. A proper application of the Grothendieck-Riemann-Roch theorem is likely to be helpful for these computations.

[‡]Such phenomena appeared in tachyons five-point computations of [7].

Finally, we hope that the ideas used in the calculations that we made will be useful for the complete study of topological $2d$ string theory.

Acknowledgments We would like to thank J. Bernstein and R. Plesser for helpful discussions.

References

[1] A. Hanany, Y. Oz and R. Plesser, "Topological Landau-Ginzburg Formulation and Integrable Structure of 2d String Theory,"hep-th/9401030, *Nucl. Phys.* **B425** (1994) 150.

[2] D. Ghoshal, S. Mukhi, "Topological Landau-Ginsburg Model of Two-Dimensional String Theory," hep-th/9312189, *Nucl. Phys.* **B425** (1994) 173.

[3] Y. Lavi, Y. Oz and J. Sonnenschein, "$(1, q = -1)$ Model as a Topological Description of $2d$ String Theory," hep-th/9406056, *Nucl. Phys.* **B431** (1994) 223.

[4] D. Ghoshal, C. Imbimbo and S. Mukhi, "Topological 2D String Theory: Higher-genus Amplitudes and W-infinity Identities," MRI-PHY/13/94, CERN-TH-7458/94, TIFR/TH/39-94, hep-th/9410034.

[5] E. Witten, "The N Matrix Model and Gauged WZW Models," *Nucl. Phys.* **B371** (1992) 191.

[6] S. Mukhi and C. Vafa, "Two Dimensional Black Hole As a Topological Coset Model of $c = 1$ String Theory," hep-th/9301083, *Nucl. Phys.* **B407** (1993) 667.

[7] R. Dijkgraaf, G. Moore and R. Plesser, unpublished.

[8] A. Lossev, "Descendants Constructed From Matter Fields in Topological Landau-Ginzburg Theories Coupled to Topological Gravity," ITEP Preprint PRINT-92-0563 (Jan 1993), hepth/9211089.

[9] R. Dijkgraaf, G. Moore and R. Plesser, "The partition function of 2D string theory," hep-th/9208031, *Nucl. Phys.* **B394** (1993) 356.

[10] E. Witten, "Two Dimensional Gravity and Intersection Theory on Moduli Space," In Cambridge 1990, Proceedings, Surveys in Differential Geometry 243-310.

[11] M. Kontsevich, "Intersection Theory on The Moduli Space of Curves and The Matrix Airy Function," *Comm. Math. Phys.* **147** (1992) 1.

[12] E. Witten, "Algebraic Geometry Associated With Matrix Models of Two Dimensional Gravity", IASSNS-HEP-91/74.

[13] P. di Francesco, D. Kutasov, "World Sheet and Space-Time Physics in Two Dimensional (Super) String Theory", hep-th/9109005, *Nucl. Phys.* **B375** (1992) 119.

[14] A. Hanany and Y. Oz, "$c = 1$ Discrete States Correlators via $W_{1+\infty}$ Constraints," hep-th/9410157, To appear in *Phys.Lett.B.*